JOINT AZIMUTH-ELEVATION-FREQUENCY ESTIMATION BY EXPLOITING BOTH SIDES OF SVD

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ABSTRACT

The problem of three-dimensional (3-D) estimation (azimuth, elevation and carrier) of multiple planes waves incident on a uniform cross array of sensors is considered in this paper. The proposed algorithm exploits both left and right singular vectors in the singular value decomposition (SVD) of arriving signals. Azimuth and elevation angles are estimated using the left singular vector information while carrier frequency estimates can be found from the right singular vector information. Automatic pairing of the spatial-temporal frequency estimates is achieved as they are associated with the same singular values.

1. INTRODUCTION

Among signal parameter estimation techniques, ESPRIT [1] algorithm has been attracting considerable attention due to their simplicity and high-resolution capability. ESPRIT kept evolving during the past two decades. The theory has been enriched and successful applications have been reported.

Main problem of joint estimation is how to automatically obtain a correct match pair. The automatic pairing can be found in two-dimensional (2-D) Unitary ESPRIT [2]. However, the algorithm becomes more sophisticated for higher dimensional cases. The algorithm based on joint diagonalization has been reported in [3][4]. In [5], real-time 3-D estimation based on PRO-ESPRIT has been proposed. To solve the complicated problem of severe under-sampling in both the temporal and spatial domain, Unitary ESPRIT has been extended in [6] to the R-dimensional case based on a simultaneous Schur decomposition. The 3-D total least squares (TLS) ESPRIT with phase averaging has been discussed in [7]. The most recent result in joint angle-frequency estimation using ESPRIT has also been given in [8].

In this paper, we simple extend 2-D Unitary ESPRIT to 3-D cases for solving the problem of joint frequency and 2-D arrival angle estimation. The algorithm exploits both left and right singular vectors in the singular value decomposition (SVD) of arriving signals. Therefore, automatic pairing of frequencies and angles is achieved without any effort.

2. PROBLEM FORMULATION

Consider \( d \) narrow-band noncoherant plane waves impinging on a 2-D centro-symmetric cross array with an \( M_x \) element linear array (ULA) aligned on \( x \)-axis and an \( M_y \) element ULA aligned on \( y \)-axis. Note here that the algorithm can be directly applied to the uniform rectangular array case as discussed in [2]. For \( N' \) snapshots, the outputs of \( x \)-array, \( X_x \), and \( y \)-array, \( X_y \), can be stacked into an \( M \times N' \) matrix as

\[
X \triangleq \begin{bmatrix} X_x \\ X_y \end{bmatrix} = A(\phi, \theta)S + N
\]

where \( M = M_x + M_y \), \( S \) is the \( d \times N' \) matrix of source waveforms, and \( N \) is the \( M \times N' \) matrix of sensor noise. The matrix \( A(\phi, \theta) \) represents an \( M \times d \) steering matrix as a function of azimuth (\( \phi \)) and elevation (\( \theta \)) angles. Let \( \Delta \) and \( \lambda_i \) denote the interelement spacing of the sensor array and the wavelength of the \( i \)-th sources, respectively. The steering matrix \( A \) can be described as

\[
A \triangleq \begin{bmatrix} A_x \\ A_y \end{bmatrix} = \begin{bmatrix} a_{x1} & a_{x2} & \cdots & a_{xd} \\ a_{y1} & a_{y2} & \cdots & a_{yd} \end{bmatrix}
\]

where

\[
\begin{align*}
a_{xi} &= [e^{-j(M_x - 1)\mu_i} \cdots e^{-j\mu_i} 1 e^{j\mu_i} \cdots e^{j(M_x - 1)\mu_i}]^T \\
a_{yi} &= [e^{-j(M_y - 1)\nu_i} \cdots e^{-j\nu_i} 1 e^{j\nu_i} \cdots e^{j(M_y - 1)\nu_i}]^T \\
\mu_i &= \frac{2\pi}{\lambda_i} \Delta \cos \phi_i \sin \theta_i, \quad \nu_i = \frac{2\pi}{\lambda_i} \Delta \sin \phi_i \sin \theta_i.
\end{align*}
\]

In order to estimate the frequency, the output of each sensor is sampled in pairwise fashion with the delay \( \delta \) such that two interleaved subspaces of signal are formulated. The time-delay \( \delta \) is chosen to be less than or equal to the Nyquist sampling interval, e.g., for 0-1 GHz signals, we have \( \delta \leq 0.5 \text{ ns} \).

Consider the \( d \) sources of interest represented by complex exponential \( s_i(n) = s_i(n)e^{2\pi f_in} \) for \( i = 1, \ldots, d \).

Let \( \Omega \) be the set of all possible frequency-azimuth-elevation patterns. Therefore, the problem of joint estimation is equivalent to finding the set of indices \( \Omega \) and corresponding frequency, azimuth, and elevation angles.
with complex amplitude $s_i(n)$ and frequency $f_i$. For $N' = 2N$ snapshots, the $d \times 2N$ matrix of source waveforms $S$ can be modeled as

$$S = [s_1 \ s_2 \ \ldots \ s_d]^T$$  \hspace{1cm} (4)

where

$$s_i = [s_i(0) \ s_i(\delta) \ s_i(1+\delta) \ \ldots \ s_i(N-1) \ s_i(N-1+\delta)].$$  \hspace{1cm} (5)

We further assume that the signals are narrow band, i.e., $s_i = s_i(0) \approx s_i(\delta) \approx \ldots \approx s_i(N-1+\delta)$ for the $i$th source. Therefore, the matrix of source waveforms $S$ can be written as

$$S = \mathbf{B}$$  \hspace{1cm} (6)

where $\mathbf{B} = \text{diag}(\mathbf{b})$ denotes the diagonal matrix with elements of the vector $\mathbf{b} = [b_1 \ b_2 \ \ldots \ b_d]^T$ being its main diagonal elements and $\mathbf{B}$ is the matrix of complex exponential terms representing carrier frequency, i.e., $b_i(n) = e^{j2\pi f_i n}$. In particular, we have

$$\mathbf{B} = [b_1 \ b_2 \ \ldots \ b_d]^T$$  \hspace{1cm} (7)

where

$$b_i = [b_i(0) \ b_i(\delta) \ b_i(1+\delta) \ \ldots \ b_i(N-1) \ b_i(N-1+\delta)].$$  \hspace{1cm} (8)

In summary, the data matrix $X$ is now modeled as

$$X = \begin{bmatrix} X_x \\ X_y \end{bmatrix} = \begin{bmatrix} A_x(\phi, \theta) \\ A_y(\phi, \theta) \end{bmatrix} \mathbf{B}(\omega) + \mathbf{N}.$$  \hspace{1cm} (9)

This signal model can be applied to any ESPRIT-like algorithms for solving joint angle-frequency estimation problem. Next section, we describe the extension of the 2-D angle estimation technique using the Unitary ESPRIT to obtain azimuth-elevation angle and carrier estimates.

### 3. 3-D UNITARY ESPRIT

Unitary ESPRIT [9][2] is a low-complexity modification of conventional ESPRIT formulated in terms of real-valued computations. In this paper, we extend Unitary ESPRIT to solve joint frequency-angle estimation problem by exploiting both left and right singular vectors of the SVD of the matrix $X$ in (9). To obtain real-valued computations, unitary matrices, $Q_{Mx}$, $Q_{My}$ and $Q_{2N}$ [2], are multiplied on both sides of the received data matrix $X$ as

$$Y = \begin{bmatrix} Q_{Mx}^H \\ 0 \\ 0 \\ Q_{My}^H \end{bmatrix} X Q_{2N}$$  \hspace{1cm} (10)

We then formulate the real-valued matrix $Z$ as

$$Z = \begin{bmatrix} \Re\{Y\} \\ \Im\{Y\} \end{bmatrix}.$$  \hspace{1cm} (11)

Consequently, the SVD of $Z$ is

$$Z = E\Sigma F^T$$  \hspace{1cm} (12)

where $E$ is a $2M \times 2M$ matrix of left singular vectors, $F$ is a $4N \times 4N$ matrix of right singular vectors and $\Sigma$ is a $2M \times 4N$ matrix consisting of singular values on main diagonal ordered in descending magnitude.

#### 3.1. 2-D Angle Estimation

Beginning with the case of a single source, the array manifold in (3) satisfies the invariance relation

$$\begin{bmatrix} e^{j\mu_1} \\ 0 \end{bmatrix} [J_{x1} \ 0] [a_{x1}]^T = \begin{bmatrix} J_{x2} \ 0 \end{bmatrix} [a_{x1}]$$  \hspace{1cm} (13)

where $J_{x1}$, $J_{x2}$, $J_{y1}$, and $J_{y2}$ are the selecting matrices. The matrices $J_{x1}$ and $J_{x2}$ select the first and last $Mx - 1$ components of an $Mx \times 1$ vector, respectively. As discussed in [2], the invariance relationship of the transformed manifold is

$$\tan\left(\frac{\mu_1}{2}\right) K_{x1} d_{x1} = K_{x2} d_{x1}$$  \hspace{1cm} (14)

for $x$-array, where a real-valued manifold $d_{x1} = Q_{Mx}^H a_{x1}$ and

$$K_{x1} = \Re\{Q_{Mx}^H J_{x1} Q_{Mx}\}$$  \hspace{1cm} (15)

$$K_{x2} = 2 \Im\{Q_{Mx}^H J_{x2} Q_{Mx}\}.$$  \hspace{1cm} (16)

Similarly, for $y$-array, we have

$$\tan\left(\frac{\nu_1}{2}\right) K_{y1} d_{y1} = K_{y2} d_{y1}$$  \hspace{1cm} (18)

where a real-valued manifold $d_{y1} = Q_{My}^H a_{y1}$ and

$$K_{y1} = 2 \Im\{Q_{My}^H J_{y1} Q_{My}\}$$  \hspace{1cm} (19)

$$K_{y2} = \Re\{Q_{My}^H J_{y2} Q_{My}\}.$$  \hspace{1cm} (20)

For the case of $d$ sources impinging on the cross array, the relation (14) and (16) are extended to

$$K_{x1} D_x \Theta_x = K_{x2} D_x, \quad \text{and} \quad K_{y1} D_y \Theta_y = K_{y2} D_y$$  \hspace{1cm} (18)

where

$$D_x = [d_{x1} \ \ldots \ d_{xd}], \quad D_y = [d_{y1} \ \ldots \ d_{yd}]$$

$$\Theta_x = \text{diag}\left(\begin{bmatrix} \tan\left(\frac{\mu_1}{2}\right) \\ \ldots \\ \tan\left(\frac{\mu_d}{2}\right) \end{bmatrix}\right)$$

$$\Theta_y = \text{diag}\left(\begin{bmatrix} \tan\left(\frac{\nu_1}{2}\right) \\ \ldots \\ \tan\left(\frac{\nu_d}{2}\right) \end{bmatrix}\right).$$  \hspace{1cm} (19)

Let $E_s$ be an $M \times d$ real-valued matrix of left singular vectors $E$ corresponding to its $d$ dominant singular values. The matrix $E_s$ is related to the real-valued manifold $D_x$ and $D_y$ as

$$E_s = \begin{bmatrix} E_{sx} \\ E_{sy} \end{bmatrix} = \begin{bmatrix} D_x \\ D_y \end{bmatrix} T_a.$$  \hspace{1cm} (20)
Substitute (20) into (18), we obtain the least squares problems
\[
K_{x1} E_x \Phi_x = K_{x2} E_x
\]
\[
K_{y1} E_y \Phi_y = K_{y2} E_y
\]
(21)

where
\[
\Phi_x = T_a^{-1} \Theta_x T_a
\]
\[
\Phi_y = T_a^{-1} \Theta_y T_a.
\]
(22)

Since all of the quantities in (21) and (22) are real-valued, automatic pairing of \( \mu \) and \( \nu \) spatial frequency estimates is achieved by computing the eigen decomposition of the matrix.
\[
\Phi_x + j \Phi_y = T_a (\Theta_x + j \Theta_y) T_a^{-1}.
\]
(23)

Azimuth angles \( \phi_i \) and elevation angles \( \theta_i \), for \( i = 1, \ldots, d \), can be calculated from
\[
\phi_i = \frac{180}{\pi} \tan^{-1} \left( \frac{\tan^{-1}(\Theta_y[i,i])}{\tan^{-1}(\Theta_x[i,i])} \right)
\]
\[
\theta_i = \frac{180}{\pi} \sin^{-1} \left( \frac{\lambda_i}{2\pi \Delta} \sqrt{\left(\tan^{-1}(\Theta_x[i,i])\right)^2 + \left(\tan^{-1}(\Theta_y[i,i])\right)^2} \right)
\]
(24)

where \( [\Theta_x]_{ii} \) and \( [\Theta_y]_{ii} \) are the diagonal elements of \( \Theta_x \) and \( \Theta_y \), on the other hand, the eigenvalues of \( \Phi_x \) and \( \Phi_y \), respectively. The wavelength \( \lambda_i \) of the \( i \)-th source can be obtained from the carrier frequency estimates described next section.

3.2. Carrier Frequency Estimation

A time-window frequency vector of complex exponentials \( b_i \) in (8) satisfied an invariance relation
\[
e^{2\pi f_i \delta} J f_1 b_i = J f_2 b_i
\]
(25)

where \( J f_1 \) and \( J f_2 \) are the \( N \times 2N \) matrices selecting alternate column of the signal \( b_i \).

The real-valued matrix of complex exponentials can be manipulated in the same manner as in 2-D angle estimation case by pre-multiplying the vector \( b_i \) by the unitary matrix \( Q_{2N}^H \) and obtained \( d_{f_1} = Q_{2N}^H b_i \). The invariance relation of the transformed complex exponential matrix \( D_f \) can be described as
\[
K_{f1} D_f \Omega = K_{f2} D_f
\]
(26)

where
\[
D_f = [d_{f1} \ldots d_{fd}]
\]
\[
K_{f1} = \Re \{ Q_{2N}^H J f_1 Q_{2N} \},
\]
\[
K_{f2} = \Im \{ Q_{2N}^H J f_2 Q_{2N} \}
\]
\[
\Omega = \text{diag} \left( [\tan(\pi f_1 \delta) \ldots \tan(\pi f_d \delta)] \right).
\]
(27)

Carrier frequency of the signals can be estimated using \( F_s \), a \( 2N \times d \) real-valued matrix of right singular vectors \( F \) corresponding to its \( d \) dominant singular values. In analogous the angle estimation previously described, the invariance relation of two subspaces shifted in time domain is
\[
( K_{f1} F_s ) \Psi = K_{f2} F_s.
\]
(28)

The carrier frequency can be obtained from the eigenvalues of the \( d \times d \) solution \( \Psi \) to the \( N \times d \) matrix equation in (28), i.e.
\[
\Psi = T_f \Omega T_f^{-1}
\]
(29)
and
\[
f_i = \left| \tan^{-1}(\Omega_{ii}) \right| / \pi \delta
\]
(30)

where \( [\Omega_{ii}] \) is the diagonal element of the real-valued matrix \( \Omega \), the eigenvalues of \( \Psi \). Automatic pairing of the spatial-temporal frequency estimates is achieved since they are associated with the same singular values. The proposed 3-D Unitary ESPRIT for solving the problem of joint frequency and 2-D arrival angle estimation is summarized below.

3.3. Summary of 3-D Unitary ESPRIT for joint frequency and 2-D arrival angle estimation

1. Compute the real value matrix \( Z \).
2. Compute the SVD of \( Z \) to obtain the \( d \) "largest" left singular vector \( E_s \) and the \( d \) "largest" right singular vector \( F_s \).
3. Compute \( \Psi \) in (28).
4. Compute \( \Omega \) as the matrix with the eigenvalues of the matrix \( \Psi \) being its main diagonal elements.
5. Compute temporal frequency estimates \( f_i \) using (30).
6. Compute \( \Phi_x \) and \( \Phi_y \) in (21).
7. Compute \( \Theta_x \) as the matrix with the eigenvalues of the matrix \( \Phi_x + j \Phi_y \) being its main diagonal elements.
8. Compute spatial frequency estimates \( \phi_i \) and \( \theta_i \), \( i = 1, \ldots, d \) using the frequency estimates \( f_i \) and (24).

4. SIMULATION EXAMPLE

In the simulation example, we consider the same system as given in [5] where the intermediate frequency (IF) signals with 1 GHz bandwidth is sampled at a rate of 250 MHz. The time-delay \( \delta \) is chosen to be equal to the Nyquist sampling interval, i.e., \( \delta = 0.5 \) ns. We assumed two IF signals with the frequencies of 500 MHz and 100 MHz impinging on a uniform cross array of antennas (\( \Delta = \lambda_{1G} / 2 \)) at the azimuth angles of 75 and -40 degree and the elevation angles of 40 and 20 degree, respectively. For \( M = 12 \) and \( N = 100 \), the errors of estimation computed from 250 independent
The problem of three-dimensional (3-D) estimation (azimuth, elevation and carrier) of multiple plane waves incident on a uniform cross array of sensors is considered in this paper. The design example is based on a Unitary ESPRIT [2]. We consider the system having a temporal undersampling technique for the frequency estimation, whereas a standard uniform spatial sampling is used for the angle estimation. The algorithm exploits both left and right singular vectors in the singular value decomposition (SVD) of arriving signals. Azimuth and elevation angles are estimated using the left singular vector information while carrier frequency estimates can be found from the right singular vector information. Automatic pairing of the spatial-temporal frequency estimates is achieved as they are associated with the same singular values.

5. CONCLUSION

6. REFERENCES


